2017 SUnMaRC Problem Contest

List of Problems

Please read the **2017 SUnMaRC Problem Contest Rules and Guidelines**.

Problems for Sessions 1A and 1B

1. (Three-Pile NIM) This problem is related to Dr. Wilson's plenary talk *To Fight And Never Lose*. The rules for NIM are in the packet at your table.

   (1) Three piles, two players.

   (2) Take as many as you like from any one pile or the same number from each of the three piles.

   (3) The last player to be able to play, wins!

   You are to play from a position where the piles have 6, 3, 2 pebbles, respectively. What is your move and why?

2. (Numerate) This problem is related to Dr. Wilson's plenary talk *To Fight And Never Lose*. The rules for Numerate are in the packet at your table.

   (1) Board is 5×5, counters $1, 2, 3, \ldots 25$.

   (2) Counters are played on the board in order $1, 2, 3, \ldots$

   (3) First player plays 1 anywhere on the board.

   (4) In each turn, the counter goes in the same row or column as the previous, with no counter between.

   (5) Last player able to play, wins!

   Who will win on a 5×5 board and how?

3. (Sorcerer's Apprentice) This problem is related to Dr. Wilson's plenary talk *To Fight And Never Lose*. The rules for Sorcerer's Apprentice are in the packet at your table.

   (1) Two players, Rows (who plays first) and Columns (who play second).

   (2) An $m \times n$ board and mn counters labelled 1 through mn.

   (3) On your turn, put one counter on the board.

   (4) When the board is full, Rows adds up each row and scores the difference between highest and lowest.

   (5) When the board is full, Columns adds up each column and scores the difference between highest and lowest.

   (6) High score Wins!

   In Sorcerer's Apprentice, suppose the first three moves are:

4		
	5	
		6

With best play, who should win and why?

4. (Elastic NIM) This problem is related to Dr. Wilson's plenary talk *To Fight And Never Lose.*

   (1) Two players and a pile of pebbles.

   (2) On the first turn, the player may take any number except that he cannot take them all.

   (3) On each succeeding turn the play may take up to twice as many as the previous player took.

   (4) The last to able to play, wins!

   You are first player in a game with 17 pebbles. What is your move and why?

5. The shoelace of the Green Giant's sneaker is 2m long. We mark the points that divide the shoelace into three equal pieces. Then we mark the points that divide the shoelace into four equal pieces. Finally we mark the points that divide the shoelace into five equal pieces. How many distinct lengths do the pieces have if we cut the string at every marking point?

6. The 7-digit number $\overline{x20y17z}$ is divisible by 792. What is the value of $x + y + z$?

7. Three actors and their three agents want to cross a river in a boat that is capable of holding only two people at a time, under the constraint that no actor can be in the presence of another agent unless their own agent is also present, because each agent is worried their rivals will poach their client. How should they cross the river with the least amount of rowing?

8. Evaluate $\lim_{x\to\infty} x \int_0^x e^{t^2-x^2}\,dt$.

9. Find $f^{(26)}(0)$, where f is defined by $f(x) = \begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$.

10. Show that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^{\ln(n)}}$ converges.

11. Show that $\sum_{n=2}^{\infty} \frac{1}{\ln(n)^{\ln(\ln(n))}}$ diverges.

12. What is the probability that a friend's phone number contains a 2? Phone numbers contain seven digits and cannot start with a 1.

13. Find $\det(A)$ if $A = \begin{bmatrix} -8 & -7 & & \cdots & -7 \\ -7 & -8 & -7 & \cdots & -7 \\ \vdots & & \ddots & & \\ -7 & \cdots & -7 & -8 & -7 \\ -7 & \cdots & & -7 & -8 \end{bmatrix} \in \mathbb{R}^{72\times 72}$.

14. If x and y are real numbers with $x^2 + y^2 = 3xy$, compute $\dfrac{x+y}{x-y}$.

15. The VIP cafeteria door on the Death Star promptly opens at 11:00 am and closes at 1:00 pm (Standard Galactic Time). Nobody is allowed to enter at other times but guests can stay until they finish their meal. To keep their lean physiques, Sith Lords usually spend their allotted 13-minute lunch break in the cafeteria sipping organic kale smoothies. Darth Sidious has a yoga class at 11:00 am, so he never has lunch before noon. Darth Vader must use a straw, so he is allowed an additional 2 minutes to slurp his smoothie. What is the probability that the two of them meet today in the cafeteria?

16. Find a matrix B such that $ABA = A$ where $A = \begin{bmatrix} -5 & 3 & 1 \\ 3 & -1 & 1 \\ -4 & 1 & -2 \end{bmatrix}$.

17. Our space ship is at a Star Base with coordinates $(1, 1)$. Our hyper drive allows us to jump from coordinates (a, b) to either coordinates $(a, a + b)$ or to coordinates $(a + b, b)$. Which points in the plane can we get to by using our hyper drive?

18. Suppose you have 9 coins, all identical in appearance and weight except for one that we know is heavier than the other 8 coins. What is the minimum number of weighings one must do with a two-pan scale in order guarantee that we can identify the counterfeit coin.

19. What is the largest power of 9 that divides 2017 factorial?

20. Two sides of a triangle has side lengths 3 and 5. The perimeter of the triangle is an integer that is divisible by 3. What is the third side of the triangle?

21. Consider the inner product $\langle p(x), q(x)\rangle := \int_0^1 p(x)q(x)dx$ on the vector space V of real polynomials with degree less than 2. The linear functional $f : V \to \mathbb{R}$ is defined by $f(p(x)) = p'(16) + 7p(-2)$. Find an $r(x) \in V$ such that $f(p(x)) = \langle p(x), r(x)\rangle$ for all $p(x) \in V$.

22. Find the matrix of the reflection $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the plane spanned by $(-2, -7, -8)$ and $(6, 6, 9)$.

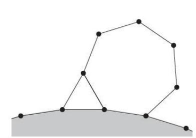
23. A square matrix is magic if the sum of the numbers in each row, column and main diagonal is the same. Let V be the vector space of 3×3 magic squares. Find the characteristic polynomial of $T : V \to V$ defined by $T(M) = M^T$.

24. There are 97 mosquitos sitting on a 3m $\times$ 2m rectangular shaped picnic table. Prove that we can eliminate 2 of the mosquitos at the same time with a 25cm $\times$ 25cm slapper.

25. The side lengths of a pentagon are 6, 5, 3, 3, 4 in this order. The pentagon has an incircle. Every side of the pentagon is divided into two pieces at the point where they touch the incircle. How long are these two pieces on the side with length 6?

26. Find a polyhedron with as few faces as possible and a face that has 12 sides?

27. The figure shows the joining of three regular polygons. How many vertices does the grey polygon have?



28. Let $X = \{1, 2, \ldots, 2017\}$ and $\mathcal{A}$ be the set of nonempty subsets of X. Find

$$\sum_{A\in\mathcal{A}} \left(\prod_{x\in A} x\right)^{-1}.$$

29. WITHDRAWN

30. Find a_{2017} if $a_1 := 1$ and $a_n := n(a_1 + a_2 + \cdots + a_{n1})$.

31. The side lengths a, b, c of a triangle are three consecutive numbers of a geometric sequence. Show that $a^2 + c^2 < 3ac$.

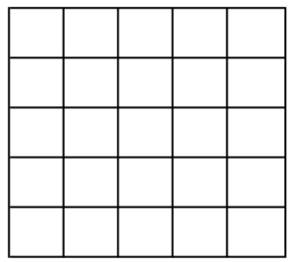
32. Consider a gambler who tosses a coin 6 times, and if it comes out Heads, wins a dollar, and if it comes out Tails, loses a dollar. She is kicked out as soon as she is in the red, i.e., has negative capital. In how many ways can she survive to 6 rounds, but at the end break even?

33. Let p_n be the the n-th prime ($p_1 = 2$, $p_2 = 3$, ...). Show that

$$\frac{1}{p_1 p_2} + \frac{1}{p_2 p_3} + \cdots + \frac{1}{p_n p_{n+1}} < \frac{1}{3}.$$

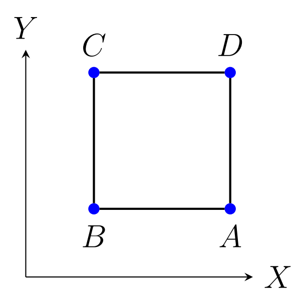
34. Find all integers n such that $n^2 + 3n + 24$ is a perfect square.

35. How many ways are there to place the letters A, B, C, D, E into the grid below, one per box, so that each letter appears exactly once in each row and in each columns and no letter appears more than once on *any* diagonal?



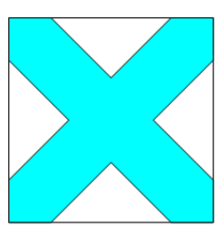
36. Let x and y be two two-digit positive integers such that y is obtained by reversing the digits of x. If x and y satisfy $x^2 - y^2 = m^2$ for some positive integer m, find x and y.

37. Square $ABCD$ has side length 6 and the side AB is parallel to the x-axis. The points A, B, and C are on $y = \log_a(x), y = 2\log_a(x)$, and $y = 3\log_a(x)$ respectively. Find a.

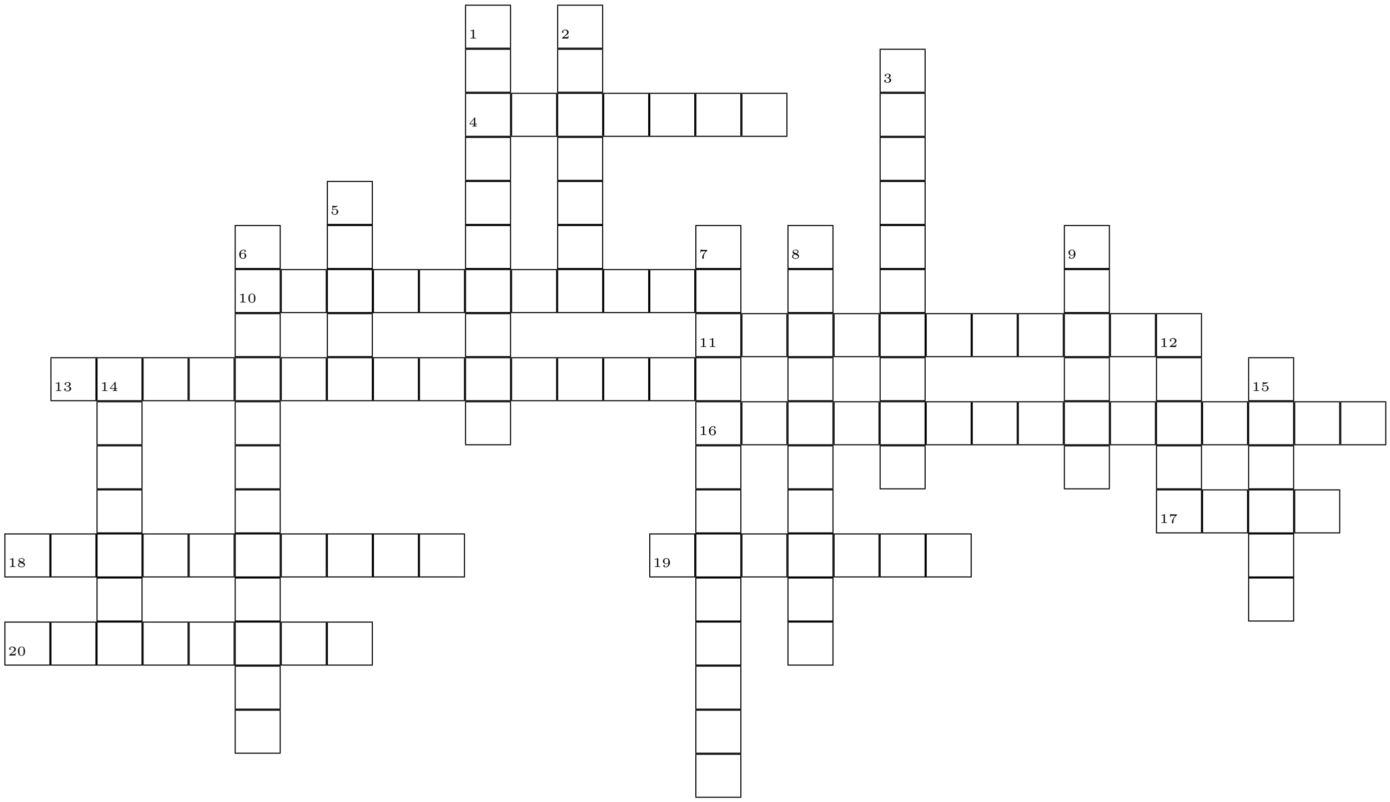


38. A box contains 2 pennies, 4 nickles, and 6 dimes. Six coins are taken out of the box randomly. What is the probability that the value of these six coins is at least 50 cents?

39. The symmetric figure shown is made by painting along two diagonals of a square with a paintbrush. Half the area of the square is painted. What is the ratio of the width of the paintbrush to the side length of the square?



Crossword 1.



Across

4 The second derivative of this function is negative
10 The chain rule is used to find the derivative of a — of two functions
11 The derivative of a function can be calculated by finding the limit of the slopes of the — (two words)
13 The second derivative is 0 here (two words)
16 $\frac{s(b)-s(a)}{b-a}$ is the — between a and b (two words)
17 The derivative of an odd function is —
18 f' is the — of f
19 If $f(a) \geq f(b)$ for all a then a is a — of f
20 Rate of change

Down

1 f is — if $a < b$ implies $f(a) < f(b)$
2 This line can be used to approximate values of a function
3 The slope of this line is 0
5 The — of a function at a point may exist even though the function is not defined there
6 The derivative of velocity
7 $f'(a)$ is the — rate of change of f at a
8 The derivative of this function is negative
9 If f' is increasing then f is —
12 $f'(a)$ is the — of the tangent line at a
14 The derivative of the exponential function with this base is the function itself
15 The derivative of this function is constant

2017 SUnMaRC Problem Contest

List of Problems

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Problems for Sessions 2A and 2B

40. Bradley buys a square piece of Styrofoam to make a stop sign for a school musical. The Styrofoam costs four dollars. To make the stop sign, which is a regular octagon, Bradley cut off the corners of the square. What is the cost of the discarded portion of the Styrofoam?

41. We call a natural number primeval if the sum of its digits is a prime. What is the maximum number of primeval numbers in a set of five consecutive natural numbers?

42. Find all primes a, b, c, d that satisfy $a^2 + b^2 + c^2 + d^2 = abcd + 4$.

43. Find all possible values of the real parameter a if no real solution x of the inequality

$$ax^2 + (1 - a^2)x - a > 0$$

satisfy $|x| > 2$.

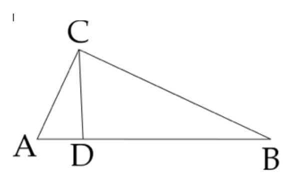
44. Determine the number of real solutions of the equation $x = 2017 \sin(x)$.

45. Let a_n be the least common multiple of the numbers in $\{1, 2, \ldots, n\}$. For example $a_1 = 1$, $a_2 = 2$, and $a_3 = 6$. Find all n such that $a_{n-1} = a_n$.

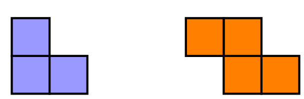
46. Let m and n be positive distinct integers such that 2017^m and 2017^n match in their last three digits. What is the minimum of $m + n$?

47. Suppose that $4^{x_1} = 5$, $5^{x_2} = 6$, $6^{x_3} = 7$, $\ldots$ $127^{x_{124}} = 128$. What is $x_1 x_2 \cdots x_{124}$?

48. In the picture below, AC is perpendicular to BC, and CD is perpendicular to AB. If $AD = 2$ and $DB = 5$, how long is CD?



49. Consider a $(2m - 1) \times (2n - 1)$ rectangular region, where m and n are integers such that $m, n \geq 4$. This region is to be tiled using tiles of the two types shown below, where each square is the same size as a square in the rectangular region. The tiles may be rotated and reflected, as long as their sides are parallel to the sides of the rectangular region. They must all fit within the region, and they must cover it completely without overlapping. What is the minimum number of tiles required to tile the region?



50. You are driving a car at constant speed around a circular racetrack. Distributed around the racetrack are cans of gasoline. The net amount of gasoline on the racetrack is precisely enough to drive the car once around the track, but the gasoline has been divided into many small containers and scattered along the course of the ride. You start with no gasoline, but as you pass each can of gasoline you (instantaneously) pick it up and put it in the gas tank. You are allowed to start your ride at any point along the track. Is it always possible to complete a full lap?

51. Consider an 8×8 grid with light-up squares. In the starting configuration, some subset of the squares are lit up. At each stage, a square lights up if at least two of its immediate neighbors (horizontal or vertical) were "on" during the previous stage. It's easy to see that for the starting configuration in which eight squares along a diagonal of the board are lit up, the entire board is eventually covered by "on" squares. Several other simple starting configurations with eight "on" squares also result in the entire board being covered. Is it possible for a starting configuration with fewer than eight squares to cover the entire board? If yes, find it; if no, give a proof.

52. Suppose you have four specially marked dice. Player 1 chooses one of the die and then Player 2 chooses one of the remaining dice. Both players then roll their respective die and the player who rolls a higher number wins. The faces of the dice are marked as follows:

    Die A: $1-1-1-5-5-5$

    Die B: $2-2-2-2-6-6$

    Die C: $3-3-3-3-3-3$

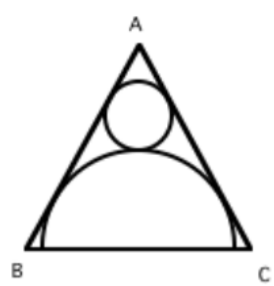
    Die D: $4-4-4-4-0-0$

    If you are Player 1, which die should you choose and why?

53. Show that if $a, b > 0$ and $a + b = 1$ then

$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}.$$

54. The figure below shows an equilateral triangle ABC with an inscribed semicircle of radius R that is tangent to sides AB and AC, and inscribed circle of radius r that is tangent to the triangle and the semicircle. Find the value of r/R.

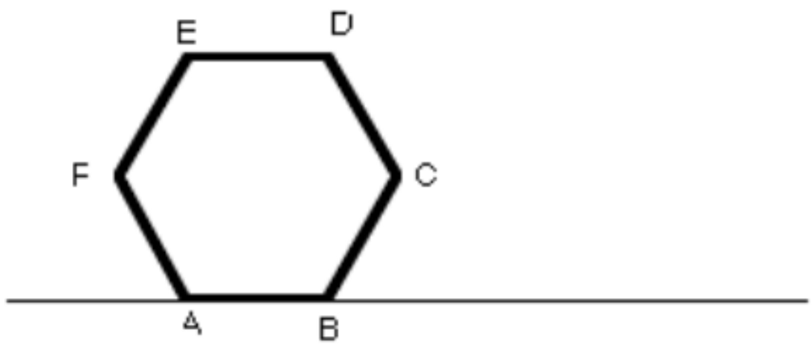


55. Find the smallest positive integer N such that $999N$ is a number made up entirely of ones.

56. In the lovely land of Ecalpon, along the southern coast, lies the Coastal Highway. Along the Coastal Highway, in order from east to west, are the villages of Abda, Bardu, Coola, Dennon, Eiran, Fiell, Gria, and Hawtch. Use the information in the partially-completely mileage chart below to determine which two of the these towns are the closest together.

Abda							
	Bardu						
		Coola					
28			Dennon				
	27			Eiran			
43		25			Fiell		
			22			Gria	
		38		24			Hawtch

57. The figure below shows a regular hexagon, 10 inches on a side, resting on a straight line. The hexagon rolls to the right without slipping until vertex A comes back in contact with the line. Find the length of the path travelled by vertex A.



58. Charles noticed that when he divided each of the numbers 291363, 415308, and 693109 by his age in years, the remainder was equal to his son's age in years every time. How old were Charles and his son at the time?

59. The sum of two positive integers is three times one of the two and it is the square of the other. Find all such pairs of numbers.

60. At MacDougall's you can order haggis MacNuggets in boxes of 6, 9, and 20. What is the largest number of nuggets that you cannot get (exactly) no matter what combination of boxes you order?

61. A large solid cube is built of identical smaller cubes such that more than half of the smaller cubes are not visible from the outside. At least how many smaller cubes are used to build the large cube?

62. What is the smallest positive integer which, when multiplied by 2001, yields a product whose last 4 digits are 2, 0, 0, 2?

63. In the game Turnaround, you are given a permutation of the numbers from 1 to n. Your goal is to get them in the natural order $1, 2, \ldots, n$. At each stage, your only option is to reverse the order of the first k places (you get to pick k at each stage). For instance, given 6375142, you could reverse the first four to get 5736142 and then reverse the first six to get 4163752. Solve the following sequence in as few moves as possible: 352614.

64. What is the smallest number which is the sum of four consecutive numbers and also the sum of five consecutive numbers? *Note:* "Number" here means positive integer.

65. Two distinct numbers, a and b, are chosen at random from the list $\{2, 2^2, 2^3, 2^4, \ldots, 2^{24}, 2^{15}\}$. What is the probability that $\log_a b$ is an integer?

66. Define a set of integers to be *rarified* provided that it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \dots, 12\}$, including the empty set, are rarified?

67. For each integer $n > 1$, let $f(n)$ be the number of solutions to $\sin(x) = \sin(nx)$ on the interval $(0, \pi)$. What is the sum $f(2) + f(4) + f(6) + \dots + f(2018)$?

68. Suppose $2^p - 1$ is a prime number. Find the sum of all positive divisors of $n = 2^{2p-1} - 2^{p-1}$.

69. Find all ordered pairs (m, n) of positive integers such that the $m^2 - n^2 = 48$.

70. Find the nonzero real solution(s) to $x = \sqrt{1 - \sqrt{1 + x}}$.

71. Consider an arithmetic sequence in which the sum of the first 100 terms is -1 and the sum of the second, fourth, . . . , and the hundredth terms is 1. Find the sum of the squares of the first 100 terms of the arithmetic sequence.

72. Find all the real solutions to the following system of equations:

$$x_1 + x_2 + \cdots + x_{2017} = 2017$$
$$x_1^4 + x_2^4 + \cdots + x_{2017}^4 = x_1^3 + x_2^3 + \cdots + x_{2017}^3$$

73. Let $f(x) = x^2 - 2ax - a^2 - 3/4$. Find all the values of a for which $|f(x)| \le 1$ on $[0, 1]$.

74. Find the maximum value of

$$\sin(x_1)\cos(x_2) + \sin(x_2)\cos(x_3) + \cdots + \sin(x_n)\cos(x_1),$$

for $n \ge 2$ arbitrary real numbers $x_1, x_2, \dots, x_n$.

75. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which $f(x + y) = f(x) + f(y)$, for all real numbers x and y.

76. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a periodic function such that

$$f(1 + x) = f(1 - x) \text{ and } f(2 + x) = -f(2 - x),$$

for all real numbers x. Find the period of f.

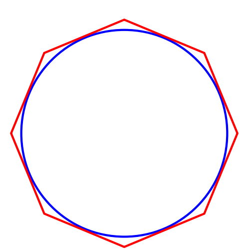
77. The polynomials P_n are defined by $P_n(x) = xP_{n-1}(x) + (1 - x)P_{n-2}(x),\ n \ge 2$ where $P_0(x) = 0$ and $P_1(x) = x$. Find all nonzero roots of $P_n(x)$ for $n \ge 3$.

78. Consider n real numbers $0 \le a_1 \le a_2 \le \cdots \le a_n$ such that

$$\sum_{i=1}^{n} a_i = 96, \ \sum_{i=1}^{n} a_i^2 = 144, \ \sum_{i=1}^{n} a_i^3 = 216.$$

Find n.

79. Consider a regular polygon with n sides circumscribed around a circle. Find the smallest integer n for which the area of the polygon is at most 101% of the area of the circle.



80. Suppose A is a 3×3 matrix with eigenvalues $1, 1$, and -1 with eigenvectors $[1, 0, 0]^T$, $[0, 1, 0]^T$, and $[1, 0, 1]^T$ respectively. Find $A^2 + A^4 + \cdots + A^{2016}$.

81. Show that when five points are placed in a 1×1 square, then there is a pair of points separated by a distance of no more than $\frac{\sqrt{2}}{2}$.

82. Consider $n+1$ distinct numbers chosen from $\{1, 2, \ldots, 2n\}$. Show that a pair of the $n+1$ numbers must be relatively prime.

83. Consider $n+1$ distinct numbers chosen from $\{1, 2, \ldots, 2n\}$. Show that for some pair of the $n+1$ numbers, one divides the other.

84. Each point of the plane is colored red or blue. Show that there is a rectangle whose vertices are all the same color.

85. Let x and y satisfy the equations $y = 2\lfloor x \rfloor + 3$ and $y = 3\lfloor x - 2 \rfloor + 5$. What is $x + y$ if x is not an integer?

86. Construct functions $f : (0, \infty) \to \mathbb{R}$ and $g : [0, \infty) \to \mathbb{R}$ that take all their output values exactly twice.

87. Let f and g be perpendicular lines in the plane. Let A and B be distinct fixed points on g but not on f. Let P be an arbitrary point on f. Let k be the line that contains A and is perpendicular to PA, and l be the line that contains B and is perpendicular to PB. Find the locus of the intersection points of k and l as P runs through f.

88. (Elastic NIM) This problem is related to Dr. Wilson's plenary talk *To Fight And Never Lose.*

    (1) Two players and a pile of pebbles.

    (2) On the first turn, the player may take any number except that he cannot take them all.

    (3) On each succeeding turn the play may take up to twice as many as the previous player took.

    (4) The last to able to play, wins!

    You are first player in a game with 17 pebbles. What is your move and why?

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$$\sum_{A \in \mathcal{A}} \left(\prod_{x \in A} x \right)^{-1}.$$

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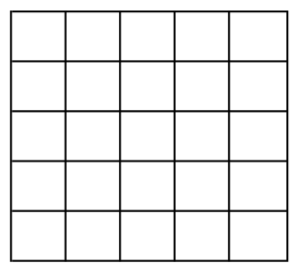

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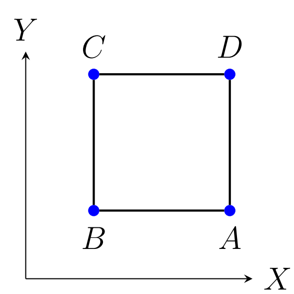
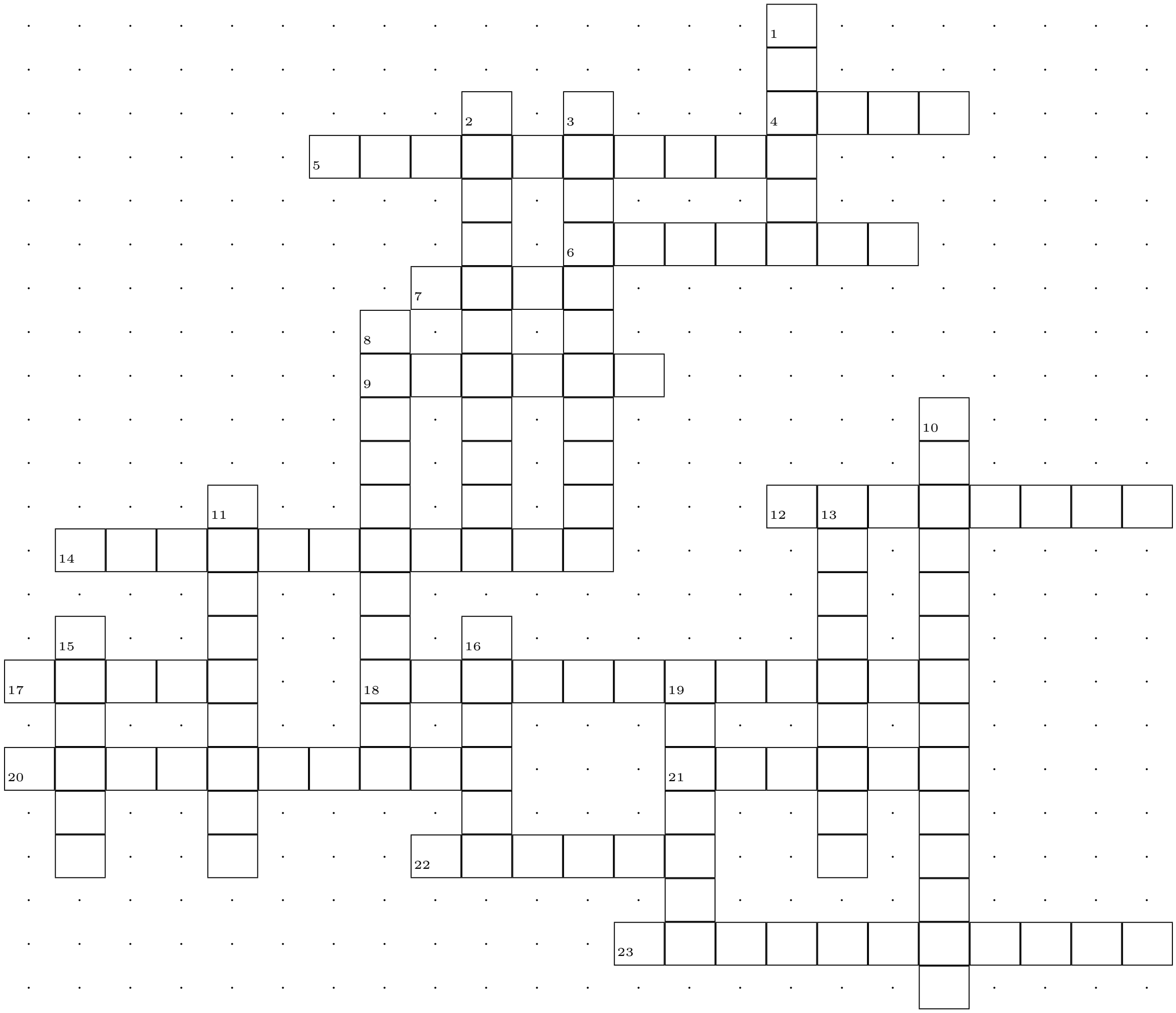


Crossword 2.



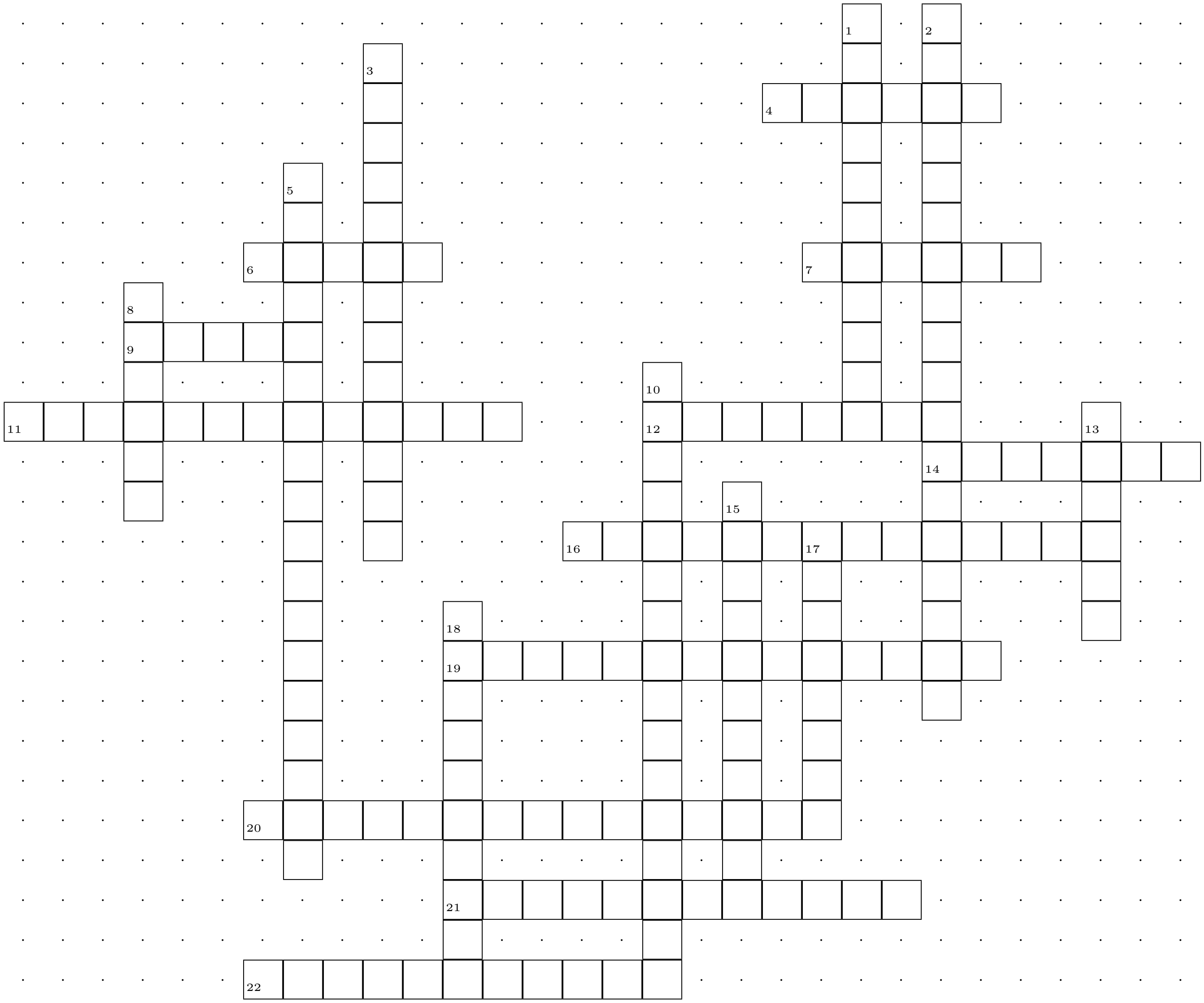
Across

4 Dimension of the row space.
5 A type of matrix.
6 A nice form of a matrix.
7 Set of linear combinations.
9 A type of transformation.
12 One is the scalar multiple of the other.
14 The scalars we need to multiply the basis vectors to get a vector.
17 Linearly independent spanning set.
18 No solutions.
20 Perpendicular.
21 An element of a vector space.
22 He has a rule.
23 A number calculated from a square matrix.

Down

1 It goes to 0.
2 Only trivial linear combination gives 0.
3 They are in the diagonal.
8 Use it to get the echelon form.
10 A polynomial.
11 Reflection of the matrix.
13 A type of matrix.
15 It has rows and columns.
16 Not a vector.
19 Multiplied by the original gives the identity.

Crossword 3.



Across

4 It is mapped to the identity element.
6 The number of cosets.
7 His theorem tells us the we only need to understand S_n.
9 The smallest power that creates the identity element.
11 Automorphisms of the regular n-gon.
12 His theorem states that something divides another thing.
14 All of its elements commute.
16 Its left and right cosets are the same.
19 The large ones are simple.
20 It has 8 elements and a bit complex.
21 A mapping.
22 Has a single generator.

Down

1 A bijection.
2 Matrices with determinant 1.
3 Formed by cosets.
5 Matrices with nonzero determinant.
8 They partition.
10 All of its elements are even.
13 Its subgroups are rarely normal.
15 Used to study field extensions.
17 Closed under multiplication and inverses.
18 Essentially the same.