Conjugate Gradient Methods for Computing Weighted Analytic Center for Linear Matrix Inequalities Under Exact and Quadratic Interpolation Line Searches

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November 16, 2021

Outline

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- Weighted Analytic Center for Linear Matrix Inequalities
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Let x, $c \in \mathbb{R}^n$, $A_i^{(j)}$ be m_j x m_j symmetric matrix.

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succeq 0 \quad (j = 1, \ldots, q) \end{array}$$

The constraints are *linear matrix inequalities* (LMIs)

Assumption: Let $\mathcal{R}$ be the feasible region

- $\mathcal{R}$ is bounded
- $\mathcal{R}$ has a nonempty interior

Example of System of LMIs

$q = 5$ LMI constraints with $n = 2$ variables:

$$A^{(1)}(x) := \begin{bmatrix} 5 & -1 \\ -1 & 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0$$

$$A^{(2)}(x) := \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix} + x_1 \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \succeq 0$$

$$A^{(3)}(x) := \begin{bmatrix} 1 & 0.5 \\ 0.5 & 2 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0$$

$$A^{(4)}(x) := \begin{bmatrix} 3.85 \end{bmatrix} + x_1 \begin{bmatrix} -0.4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \end{bmatrix} \geq 0$$

$$A^{(5)}(x) := \begin{bmatrix} 2.75 \end{bmatrix} + x_1 \begin{bmatrix} 0.8 \end{bmatrix} + x_2 \begin{bmatrix} 1 \end{bmatrix} \geq 0.$$

(Pressman & Jibrin 2001)
Consider the system of LMIs:

$$A^{(j)}(x) := A_0^{(j)} + \sum_{i=1}^{n} x_i A_i^{(j)} \succ 0, \quad (j = 1, 2, \dots, q)$$

Let $\mathcal{R}$ = feasible region. Choose $\omega > 0$.
Define the **barrier function** $\phi_\omega(x) : \mathbb{R}^n \longrightarrow \mathbb{R}$ by:

$$\phi_\omega(x) = \begin{cases} \sum_{j=1}^{q} \omega_j \text{logdet}[(A^{(j)}(x))^{-1}] & \text{if } x \in int(\mathcal{R}) \\ \infty & \text{otherwise} \end{cases}$$

The **weighted analytic center** is defined by:

$$x_{ac}(\omega) = argmin\{\phi_\omega(x) \mid x \in \mathbb{R}^n\}$$

This extends the linear case (Atkinson & Vaidya 1992)
Analytic center: when $\omega = [1, \dots, 1]$

Some Historical Facts

- Weighted analytic center for linear constraints discussed in the paper (Atkinson & Vaidya 1992)
- Infeasible Newton's method for analytic center for single LMI given in the book (Boyd & Vandenberghe 2004)
- Infeasible Newton's method for weighted analytic center for LMIs: (Jibrin 2015)

Gradient and Hessian of the Barrier Function $\phi_\omega(x)$

For $i, j = 1, \ldots, n$

$$\begin{aligned} \nabla_i \phi_\omega(x) &= -\sum_{j=1}^{q} \omega_j (A^{(j)}(x))^{-1} \bullet A_i^{(j)} \\ H_{ij}(x) &= \sum_{k=1}^{q} \omega_k [(A^{(k)}(x))^{-1} A_i^{(k)}]^T \bullet [(A^{(k)}(x))^{-1} A_j^{(k)}] \end{aligned}$$

given an interior point x, tolerance $TOL > 0$

Set $k = 1$

repeat

1. Compute the Newton's direction

$$s = -[H(x)]^{-1}\nabla\phi_\omega(x)$$

2. Compute $d = \sqrt{s^T H(x) s}$
3. Do line search to get stepsize h
4. Update $x := x + hs$
5. Update $k = k + 1$

Until $d \leq TOL$

Given: an interior point x_0 of $\mathcal{R}$, tolerance $TOL > 0$

Set $k = 1$

Repeat

1. Compute the search direction

$$d_{k+1} = \begin{cases} -g_k & \text{if } k = 0, \\ -g_{k+1} + \beta_k d_k & \text{if } k \geq 1, \end{cases}$$

3. Do line search to get stepsize α_k
4. Update $x_{k+1} = x_k + \alpha_k d_k$
5. Update $k = k + 1$

Until $\| g_k \| \leq TOL$

Conjugate Gradient Methods for Computing Weighted Analytic Center Contd

Table: Formulas for parameter β_k for methods considered

No.	β_k	Method name
1	$\frac{\|\|g_k\|\|^2}{\|\|g_{k-1}\|\|^2}$	Fletcher-Reeves(FR) method
2	$\frac{g_k^T(g_k - g_{k-1})}{\|\|g_{k-1}\|\|^2}$	Polak-Rebiere-Polyak(PR) method
3	$\frac{g_k^T(g_k - g_{k-1})}{d_k^T(g_k - g_{k-1})}$	Hestenes-Stiefel(HS) method
4	$\frac{g_k^T\left(g_k - \frac{\|\|g_k\|\|}{\|\|g_{k-1}\|\|} g_{k-1}\right)}{d_{k-1}^T(d_{k-1} - g_k)}$	(RAMI) method

Let

$$h(\alpha) = \phi_\omega(x_k + \alpha d_k)$$

The exact stepsize α_k is given by

$$\alpha_k = argmin\{h(\alpha) \mid \alpha \geq 0\}$$

Let

$$B_j(d,x) = -A^{(j)}(x)^{-\frac{1}{2}}(\sum_{i=1}^{n} d_i A_i^{(j)})A^{(j)}(x)^{-\frac{1}{2}},\ (1 \leq j \leq q)$$

Theorem: Let x_k be an interior point of $\mathcal{R}$ and $\lambda_i^{(j)}$ be the ith eigenvalue of $B_j(d_k, x_k)$. Then

$$h(\alpha) = -\sum_{j=1}^{q} \omega_j logdet(A^{(j)}(x_k)) - \sum_{j=1}^{q} \omega_j \sum_{i=1}^{m_j} log(1 + \alpha\lambda_i^{(j)}).$$

The derivatives of $h(\alpha)$ are given by

$$\begin{aligned} h'(\alpha) &= -\sum_{j=1}^{q} \omega_j \sum_{i=1}^{m_j} \frac{\lambda_i^{(j)}}{(1+\alpha\lambda_i^{(j)})} \\ h''(\alpha) &= \sum_{j=1}^{q} \omega_j \sum_{i=1}^{m_j} \left(\frac{\lambda_i^{(j)}}{1+\alpha\lambda_i^{(j)}} \right)^2 \end{aligned}$$

Iterates: $\alpha_{k+1} = \alpha_k - \dfrac{h'(\alpha_k)}{h''(\alpha_k)}$

Quadratic Interpolation

Recall that $h(\alpha) = \phi_\omega(x_k + \alpha d_k)$

Step 1: Find the distance σ_+ from x_k to the boundary of the feasible region $\mathcal{R}$ in the direction d_k

Step 2: Set $\alpha_1 = 0$ and $\alpha_3 = \sigma_+ - 0.001$

Step 3: Repeat

$\alpha_3 = \alpha_3/2$

Until $h(\alpha_3) < h(\alpha_1)$

Let $\alpha_2 = \alpha_3/2$

Quadratic Interpolation Contd

Step 4: Compute the zero α^* of the quadratic polynomial $P(\alpha)$ passing through the points $(\alpha_1, P(\alpha_1))$, $(\alpha_2, P(\alpha_2))$ and $(\alpha_3, P(\alpha_3))$.

Step 5:

if $\alpha_3 < \alpha^*$

    set $\alpha_k = \alpha_3$

else

    set $\alpha_k = \alpha^*$

end

Global Convergence

Theorem: The barrier function $\phi_\omega(x)$ is strongly convex over the interior of the feasible region $\mathcal{R}$

Remark: This guarantees FR and HS with exact line search are globally convergent

Theorem: Let x_0 be an interior point of $\mathcal{R}$. The gradient $g(x) = \nabla\phi_\omega(x)$ is Lipschitz continuous in a neighborhood of the level set $\mathcal{L} = \{x \in \mathbb{R}^n \mid g(x) \leq g(x_0)\}$.

Remark: This guarantess FR and RAMI with exact line search are globally convergent

Numerical Experiments

Test Problems: A bit random
Computer: Dell OPTIPLEX 880
Parameters: $TOL = 10^{-4}$, MaxIter=1000

LMI Test Problem	n	q	$[m_1, \ldots, m_q]$	$[\omega_1, \ldots, \omega_q]$
1	2	2	[1,2]	[4,5]
2	2	3	[5,4,5]	[3,175,1]
3	2	8	[2,4, 5, 5, 5, 1, 5, 4]	[10,10,10,1,1,1,10,1]
4	3	2	[5,4]	[100,1]
5	3	2	[3,4]	[1,10]
6	4	10	[4, 5, 1, 4, 2, 3, 5, 5, 2, 1]	[1,1,100,100,100,1,100,10,1,1]
7	4	7	[2, 4, 4, 5, 4, 2, 1]	[1,100,10,1,10,1,10]
8	5	6	[5, 1, 4, 4, 4, 5]	[10,10,1,10,1,1]
9	5	4	[4, 1, 5, 1]	[100, 1, 1,1]
10	6	3	[4, 1, 5]	[10,1,100]
11	6	8	[2, 5, 2, 5, 5, 3, 5, 2]	[1,1,1,10,10,1,100,1]
12	7	2	[5, 4]	[1,10]
13	7	4	[1, 4, 1, 2]	[1,10,1,100]

Table: Test Problems

Numerical Experiments Contd

LMI Test Problem	n	q	$[m_1, \ldots, m_q]$	$[\omega_1, \ldots, \omega_q]$
14	8	5	[1, 1, 4 , 3, 3]	[1,1,10,10,10]
15	8	5	[5, 4, 5, 2, 5]	[10,10,1,1,100]
16	9	3	[3 , 2 , 5]	[10,1,1]
17	9	3	[5 , 4, 4]	[100,100,1]
18	10	8	[4 , 2 , 3, 4 , 5, 4, 4, 2]	[10,1,1,1,10,10,1,1]
19	10	8	[4, 5, 3 , 5, 4, 2, 2, 4]	[1,10, 1,100,1,10,10,1]
20	10	9	[5 , 2, 5, 3, 2, 1, 3, 2, 2]	[1,10,10,1,1,1,1,1,100]
21	3	6	[3, 4, 1, 5, 4, 1]	[1,1,1,1,10,1]
22	5	7	[2 , 3, 5, 5, 2, 4, 2]	[10,1,1,1,10,1,1]
23	5	3	[5, 5 , 2]	[100,1,1]
24	5	9	[2 , 4, 4, 1, 4, 5, 3 , 5 , 1]	[100,1,100,100,100,10,1,1,10]
25	10	3	[1 , 5, 2]	[1,100,10]
26	5	10	[3, 4, 1, 3, 1, 4, 4, 5, 4, 4]	[7, 7, 7, 8, 7, 6, 4, $\mathbf{10^6}$, 3, 6]
27	3	7	[2, 3, 4, 5, 4, 1, 5]	[3, 5, 2, 1, $\mathbf{10^6}$, 2, 7]
28	5	7	[2 , 3, 5, 5, 2, 4, 2]	[1, 1, 4, 8, 5, 8, 3, $\mathbf{10^6}$]
29	3	8	[5, 3, 3, 5, 5, 4, 2, 3]	[2, $\mathbf{10^6}$, 4, 3]
30	2	6	[4, 4, 3, 1, 5, 2]	[4, 6, 5, 3, $\mathbf{10^6}$, 4]

Table: Test Problems

Numerical Experiments Contd

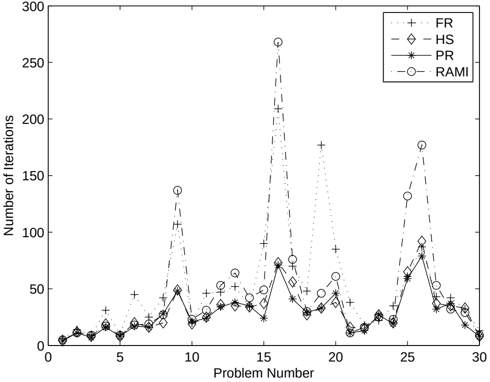

Figure: Problem Number Vs Iterations taken by each method to find the weighted analytic center using exact line search (Newton's method), where $+$=FR, $\diamond$=HS, *=PR, o=RAMI.

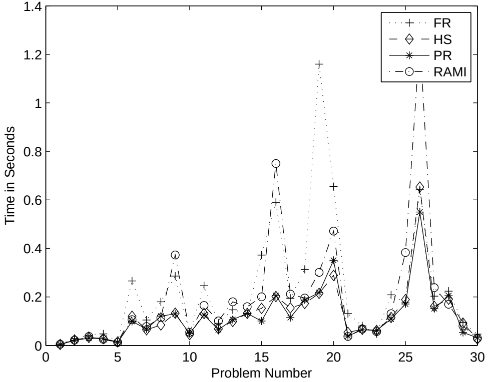

Figure: Problem Number Vs Time taken by each method to find the weighted analytic center using exact line search (Newton's method), where +=FR, ⋄=HS, *=PR, o=RAMI.

Numerical Experiments Contd

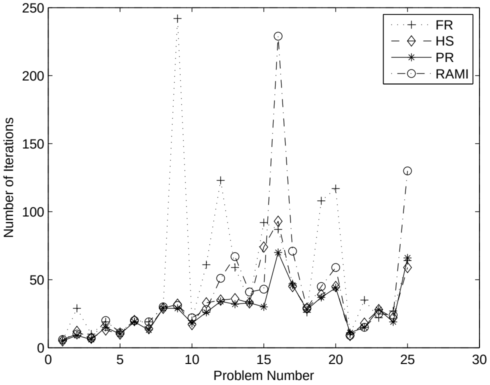

Figure: Problem Number Vs Iterations taken by each method to find the weighted analytic center using inexact line search (Quadratic Interpolation) for the 25 problems where all four methods were successful and $+$=FR, $\diamond$=HS, *=PR, o=RAMI.

Numerical Experiments Contd

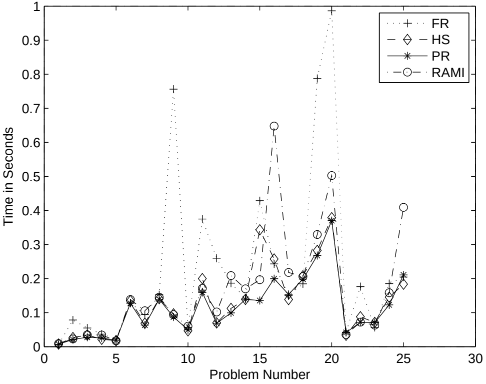

Figure: Problem Number Vs Time taken by each method to find the weighted analytic center using inexact line search (Quadratic Interpolation) for the 25 problems where all four methods were successful and $+$=FR, $\diamond$=HS, *=PR, o=RAMI.

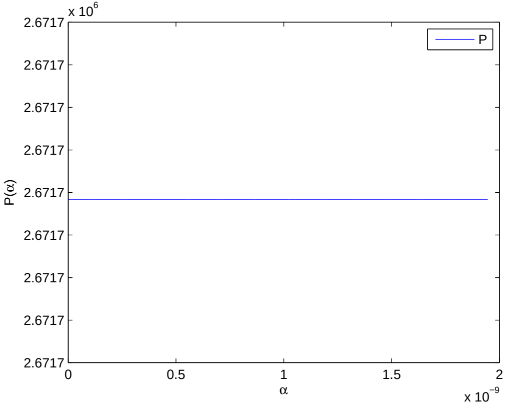

Figure: Graph of the Quadratic approximation $P(\alpha)$ for Problem 30 with $w = [4, 6, 5, 3, \mathbf{10^6}, 4]$ at iteration 12. Note $P(\alpha)$ is flat over the interval $[h_1, h_3] = [0, 1.9462x10^{-9}]$. Hence, at iteration 13, FR with Quadratic interpolation line search fails.

Summary and Conclusion

- Presented four conjugate gradient methods for weighted analytic center for LMIs - FR, HS, PR, RAMI
- Worked well with Exact line search and Quadratic interpolation line search
- PR is the best method, followed by HS, then RAMI, and then FR
- Exact line search handles weights better than the Quadratic interpolation line search when some weight is relatively much larger than the other weights
- FR is more susceptible to jamming phenomenon than both PR and HS
- PR and HS are superior to RAMI with the problems considered

Future Work

- Conjugate gradient methods for weighted analytic center for Second-order Cone Constraints
- Consider other inexact line searches for the conjugate gradient methods

THANK YOU!